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Non-linear free vibration of laminated composite plate with random material properties

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Abstract

Composite laminates are widely used in construction of mechanical, aerospace, marine and automotive structure. These structures exhibit inherent random dispersion in material properties, as absolute control of production process is neither feasible nor economical. Some composite structures are subjected to large amplitude vibration during their working life that may lead to non-linearity in the response. The present paper analyses the effect of material parameter dispersion on the large amplitude free vibration of especially orthotropic laminated composite plates. The basic formulation of the problem has been developed based on the classical laminate theory and Von-Karman non-linear strain–displacement relation. The system equations have been obtained by using Hamilton's principle and the solution has been found by term wise series integration. Perturbation technique has been used to obtain the second order response statistics. Typical results have been presented for a plate with all edges simply supported. Effects of side-to-thickness ratio, aspect ratio, oscillation amplitude and mode shape along with change in standard deviation of material properties have been investigated for cross-ply symmetric and antisymmetric laminates.

1. Introduction

Composite are being increasingly adopted in diverse engineering applications. Laminated composite plates are fast replacing metal alloys in most light transport vehicles. Many aerospace and other high-speed vehicle components are being fabricated with composites. The properties of composite display considerable scatter because of the uncertainties involved at many levels in fabrication and manufacturing—properties of its constituents, geometrical parameters of laminates, fiber orientations, volume fraction, inclusions, voids and others. It is not practically

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possible to control these variations completely, resulting in inherent variations in the material and geometric properties of the laminates. The uncertainties in the system parameters are reflected in uncertainty in its response behavior. It is important for the designer to have an accurate knowledge of the structural response for sensitive applications. Enhanced accuracy in response evaluation is possible by accounting for the dispersion in material properties in the modelling of the problem. This can be achieved by taking the material properties as random.

Structural components are generally subjected to dynamic loading in their working life. Very often these may have to perform in severe dynamic environment. Accurate evaluation of the system response demands the investigation of their non-linear free vibration behavior.

Any structural analysis problem is characterized by the following three basic aspects: material of the structure, geometry of the structure and type of loading. In a real life problem, all the three aspects are random in nature. Significant volume of literature is available for external loading as random with material properties and geometry as deterministic. Nigam and Narayanan [1] have presented various classes of problems in this area.

Some published literature is available for analysis of structure with random material properties. Ibrahim [2] has reviewed topics pertaining to structural dynamics with parameter uncertainties. Leissa and Martin [3] have analyzed the vibration and buckling of rectangular composite plates and have studied the effects of variation in fiber spacing. Shinizuka and Astill [4] have employed a numerical technique to obtain statistical properties of eigenvalues of spring supported columns with the support and axial loading along with material and geometric properties as random. The method has been used to investigate the accuracy of the perturbation approach for calculation of vibration and buckling modes. Salim et al. [5–7] have studied the statistical response of rectangular composite plates considering material properties as independent random variables (RV). The second order statistics for static deflection, natural frequency and buckling load have been studied using a first order perturbation technique (FOPT). Free vibration response has been obtained by Vaicatis [8] for beams with mass and flexural rigidity as RV. Chen and Soroka [9] have studied the response of a multi-degree-of-freedom system with random properties to deterministic excitations. The system equations have been solved by perturbation technique. The second order statistics of the system response have been investigated with variation in the system property statistics. Yadav and Verma [10] have studied the buckling response of thin cylindrical shell using classical laminate theory (CLT) and have employed the FOPT for obtaining the second order statistics of buckling loads. Gorman [11] has presented free vibration analysis of thin rectangular plates with variable edge supports using the method of superimposition. Singh et al. [12,13] have studied the initial buckling and natural frequency of cylindrical panel and composite plate with random material properties and have obtained the second order statistics of response.

All the studies of composite structures in random environment are confined to small displacements in the linear domain. To the best of the authors' knowledge, large amplitude dynamic response of composite plates with random material properties has received no attention. The design margin is small in sensitive applications. Lack of adequate knowledge of the system behavior may result in failure of the design. The objective of the present work is to incorporate the non-linear effects for free vibration analysis in the random environment as an important design information. The analysis uses CLT and Von-Karman non-linear strain–displacement relationship. Hamilton's principle has been used for developing the system equations. The second order statistics of non-linear natural frequency for composite plate with random material properties has been evaluated using FOPT. All edges are assumed simply supported (SS). The numerical results for mean and standard deviation (SD) for the non-linear natural frequency with variation in second order statistics of the material properties along with the effects of thickness and aspect ratios have been obtained for symmetric and antisymmetric cross-ply laminates.

2. Non-linear natural frequency

2.1. System energies

Consider a rectangular composite plate of constant total thickness 'h', composed of thin orthotropic layers bonded together. The origin of a Cartesian co-ordinate system is located in the central plane at the left corner with x and y axes in the plane and z-axis normal to it. The displacements of a point in the mid-plane along the x, y and z directions are denoted by u, v and w, respectively.

The stress and moment resultants per unit length are defined as [14]:

$$\begin{bmatrix} N_i \\ M_i \end{bmatrix} = \begin{bmatrix} A_{ij} & B_{ij} \\ B_{ij} & D_{ij} \end{bmatrix} \begin{bmatrix} \varepsilon_j \\ \kappa_j \end{bmatrix} \quad (i,j = 1,2 \text{ and } 6), \tag{1}$$

where A_{ij}, B_{ij} and D_{ij} are the extensional, bending-extension coupling and bending stiffness matrices, respectively. ε_j and κ_j are the mid-plane strains and curvature, respectively.

The strain energy for the plate can be written as

$$U = \frac{1}{2} \int_{-h/2}^{h/2} \int_A (\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3 + \sigma_4\varepsilon_4 + \sigma_5\varepsilon_5 + \sigma_6\varepsilon_6)\, \mathrm{d}A\, \mathrm{d}z. \tag{2}$$

The stress and moment resultants can also be defined as [14]:

$$(N_i, M_i) = \int_{-h/2}^{h/2} \sigma_i(1, z)\, \mathrm{d}z \quad (i = 1,2 \text{ and } 6). \tag{3}$$

Neglecting the transverse shear effects under the CLT formulation, $\varepsilon_3, \varepsilon_4, \varepsilon_5 = 0$. Substituting Eqs. (1) and (3) in Eq. (2), the expression for the strain energy can be written as:

$$\begin{aligned} U = \frac{1}{2} \int_0^a \int_0^b \{ & A_{11}\varepsilon_1^2 + 2A_{12}\varepsilon_1\varepsilon_2 + A_{22}\varepsilon_2^2 + 2A_{16}\varepsilon_1\varepsilon_6 \\ & + 2A_{26}\varepsilon_2\varepsilon_6 + A_{66}\varepsilon_6^2 + 2B_{11}\kappa_1\varepsilon_1 + 2B_{12}(\kappa_2\varepsilon_1 + \kappa_1\varepsilon_2) \\ & + 2B_{22}\kappa_2\varepsilon_2 + 2B_{16}(\kappa_6\varepsilon_1 + \kappa_1\varepsilon_6) + 2B_{26}(\kappa_6\varepsilon_2 + \kappa_2\varepsilon_6) \\ & + 2B_{66}\varepsilon_6\kappa_6 + D_{11}\varepsilon_1^2 + 2D_{12}\kappa_1\kappa_2 + D_{22}\varepsilon_2^2 + 2D_{26}\kappa_6\kappa_2 \\ & + 2D_{16}\kappa_6\kappa_1 + D_{66}\kappa_6^2 \}\, \mathrm{d}x\, \mathrm{d}y. \end{aligned} \tag{4}$$

The kinetic energy of the plate, neglecting in-plane inertia, is:

$$T = \left(\tfrac{1}{2}\right) \int_0^a \int_0^b \left(\sum \rho_i h_i\right) \dot{w}^2(x, y, t)\, \mathrm{d}x\, \mathrm{d}y \tag{5}$$

2.2. Non-linear strain–displacement relations

Assuming the dynamic amplitudes to be large, the following Von-Karman strain–displacement relations are used [15]:

$$\varepsilon_1 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_2 = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,$$
$$\varepsilon_6 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}. \tag{6}$$

The curvature–displacement relationship can be written as:

$$\kappa_1 = -\partial^2 w/\partial x^2, \quad \kappa_2 = -\partial^2 w/\partial y^2,$$
$$\kappa_6 = -2\partial^2 w/\partial x \partial y. \tag{7}$$

2.3. Boundary condition

The boundary condition for a plate with all sides simply supported with edges free to move in their respective normal direction are given by the following set of equations:

$$\text{Along } x = 0 \text{ and } x = a \text{ for all } y: w = 0, \quad M_x = 0, \quad v = 0, \quad N_x = 0.$$
$$\text{Along } y = 0 \text{ and } y = b \text{ for all } x: w = 0, \quad M_y = 0, \quad u = 0, \quad N_y = 0. \tag{8}$$

The following set of admissible functions satisfies the boundary conditions:

$$u = u_0(t)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \quad v = v_0(t)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b},$$
$$w = w_0(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{9}$$

2.4. System equations

Substituting Eqs. (6), (7) and (9) in Eqs. (4) and (5) and carrying out the indicated integration gives the total strain energy and kinetic energy of the system. Use of Hamilton's principle leads to two algebraic and one ordinary differential equations in terms of the displacement amplitudes u_0, v_0 and w_0.

$$T_1 u_0 + T_2 v_0 + T_3 w_0 + T_4 w_0^2 = 0, \tag{10}$$

$$T_2 u_0 + T_5 v_0 + T_6 w_0 + T_7 w_0^2 = 0, \tag{11}$$

$$\left(\sum \rho_i h_i\right)\ddot{w}_0 + T_3 u_0 + T_6 v_0 + T_8 w_0 + 2T_4 u_0 w_0 + 2T_7 v_0 w_0 + T_9 w_0^2 + T_{10} w_0^3 = 0, \tag{12}$$

where the coefficients $T_1, T_2, \ldots, T_{10}$ depend on the plate geometry, material properties and the mode shapes. Their expressions are placed in Appendix A.

Substituting u_0 and v_0 in terms of w_0 from Eqs. (10) and (11) into Eq. (12) results in a second order differential equation containing quadratic and cubic non-linear terms:

$$\left(\sum \rho_i h_i\right)\ddot{w}_0 + L_1 w_0 + L_2 w_0^2 + L_3 w_0^3 = 0, \tag{13}$$

where L_1, L_2 and L_3 are as follows:

$$L_1 = [T_8 + \{(2T_2T_3T_6 - T_3^2T_5 - T_6^2T_1)/(T_1T_5 - T_2^2)\}],$$
$$L_2 = [T_9 + \{(3T_2T_3T_7 - 3T_2T_4T_6 - 3T_3T_4T_5 - 3T_1T_6T_7)/(T_1T_5 - T_2^2)\}],$$
$$L_3 = [T_{10} + \{(4T_2T_4T_7 - 2T_5T_4^2 - T_1T_7^2)/(T_1T_5 - T_2^2)\}]. \tag{14}$$

The energy balance equation is obtained by multiplying Eq. (13) by $\dot{w}_0$ and integrating with respect to time:

$$\left(\sum \rho_i h_i\right)\dot{w}_0^2 + L_1 w_0^2 + \left(\tfrac{2}{3}\right)L_2 w_0^3 + \left(\tfrac{1}{2}\right)L_3 w_0^4 = H\,(\text{const}), \tag{15}$$

where ρ_i is the density and h_i is the thickness of the ith lamina.

At $w_0 = w_{max}$ the velocity $\dot{w}_0$ is zero. Using this condition the constant H in Eq. (15) can be obtained as:

$$H = L_1 w_{max}^2 + (\tfrac{2}{3})L_2 w_{max}^3 + (\tfrac{1}{2})L_3 w_{max}^4. \tag{16}$$

Substituting this constant H in Eq. (15) yields:

$$\left(\sum \rho_i h_i\right)\dot{w}_0^2 = L_1(w_{max}^2 - w_0^2) + \left(\tfrac{2}{3}\right)L_2(w_{max}^3 - w_0^3) + \left(\tfrac{1}{2}\right)L_3(w_{max}^4 - w_0^4) \tag{17}$$

The displacement amplitude is obtained from Eq. (17) by setting $\dot{w}_0 = 0$, which signifies the point of motion reversal. For all symmetric laminates and square antisymmetric laminate the coefficient L_2 assumes zero value. For these cases the displacement amplitude has two real and opposite roots $\pm w_{max}$. Hence the non-linear time period for such a plate can be obtained as:

$$T_{nl} = \frac{2\pi}{\omega} = 4\int_0^{\pi/2} \frac{\mathrm{d}\theta\sqrt{\sum(\rho_i h_i)}}{\sqrt{[L_1 + \frac{1}{2}L_3(1+\sin^2\theta)w_{max}^2]}}. \tag{18}$$

On simplifying, one can rewrite the above equation as:

$$T_{nl} = \frac{4\sqrt{\sum(\rho_i h_i)}}{\sqrt{L_1(1+b)}}\int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1+\alpha\sin^2\theta}}, \tag{19}$$

where $b = 0.5L_3 w_{max}^2/L_1$ and $\alpha = b/(1+b)$.

Eq. (19) is in the form of an elliptic integral, which cannot be evaluated in terms of elementary functions. An infinite series representation can be generated for the above integral by first expanding the integrands in binomial series and then using term wise integration:

$$T_{nl} = \frac{2\pi\sqrt{\sum(\rho_i t_i)}}{\sqrt{L_1(1+b)}}\left[1 - \frac{\alpha}{2^2} + \frac{\alpha^2}{4^2} - \frac{\alpha^3}{8^2} + \frac{\alpha^4}{16^2} - \ \ldots\right]. \tag{20}$$

The non-linear frequency can be expressed as:

$$\omega = 2\pi/T_{nl}. \tag{21}$$

Thus ω is a function of E_{11}, E_{22}, G_{12}, v_{12} and w_{max}.

2.5. Mean and variance of natural frequency—perturbation technique

Perturbation approach has been adopted for obtaining the second order frequency statistics with randomness in material properties of the flat plate. It is assumed that the material properties are independent RV. It is also assumed that the dispersion of each material property about its mean value is small, which is true in most engineering applications.

Any random variable may be split up as the sum of its mean and zero mean random part with generality. For example

$$b_l = b_l^d + b_l^r, \tag{22}$$

where superscript 'd' denotes the mean value, which is deterministic and 'r' denotes the superimposed zero mean random component. Let b_l be the primary RV for the system selected as the basic material properties for the present study.

Using Taylor series expansion the random part of the dependent variables can be expressed in terms of the independent variables. Assuming b_l^r to be small in magnitude, the second and higher order terms are neglected and the expression may be put as:

$$\begin{aligned} L_i^r &= \textstyle\sum_{l=1}^{p} (\partial L_i^d / \partial b_l^d) b_l^r, \\ \omega^r &= \textstyle\sum_{l=1}^{p} (\partial \omega^d / \partial b_l^d) b_l^r, \end{aligned} \tag{23}$$

where L_i^r and ω^r are the zero mean random parts of the operator L_i and the system natural frequency ω. Since dispersion about the mean is small, total random processes involved in the derivatives have been approximated by their mean values.

The elements of the extensional stiffness matrix A_{ij}, coupling matrix B_{ij} and bending stiffness matrix D_{ij} are given by [14]

$$\begin{aligned} A_{ij}^d &= \sum_{k=1}^{n} (\bar{Q}_{ij})_k^d (h_k - h_{k-1}), \quad B_{ij}^d = \left(\tfrac{1}{2}\right) \sum_{k=1}^{n} (\bar{Q}_{ij})_k^d (h_k^2 - h_{k-1}^2) \\ D_{ij}^d &= \left(\tfrac{1}{3}\right) \sum_{k=1}^{n} (\bar{Q}_{ij})_k^d (h_k^3 - h_{k-1}^3) \;\; i,j = 1, 2 \text{ and } 6, \end{aligned} \tag{24}$$

where $(\bar{Q}_{ij})_k$ is the transformed reduced stiffness of the kth lamina. In general, in-plane as well as out-of-plane loads may act on the plate.

Using Taylor series expansion, keeping only one term in the series and neglecting small quantities, following Eq. (23) one can write:

$$\begin{aligned} A_{ij}^r &= \sum_{l=1}^{p} (\partial A_{ij}^d / \partial b_l^d) b_l^r, \quad B_{ij}^r = \sum_{l=1}^{p} (\partial B_{ij}^d / \partial b_l^d) b_l^r, \text{ and} \\ D_{ij}^r &= \sum_{l=1}^{p} (\partial D_{ij}^d / \partial b_l^d) b_l^r. \end{aligned} \tag{25}$$

Using Eq. (19), the derivative of ω^{d} with respect of b_l^d can be expressed as:

$$\partial\omega^d/\partial b_l^d = \frac{\dfrac{\left(2\dfrac{\partial L_1^d}{\partial b_l^d}w_{max} + \dfrac{\partial L_3^d}{\partial b_l^d}w_{max}^3\right)}{2\sqrt{2L_1^d w_{max} + L_1^d w_{max}^3}} - \sqrt{2L_1^d w_{max} + L_3^d w_{max}^3}\dfrac{\partial}{\partial b_l^d}\left[1 - \dfrac{\alpha}{2^2} + \dfrac{\alpha^2}{4^2} - \ldots\right]}{\sqrt{2w_{max}}\sqrt{\sum\rho_i t_i}\left(1 - \dfrac{\alpha}{2^2} + \dfrac{\alpha^2}{4^2} - \ldots\right)^2}. \tag{26}$$

The partial derivative of L_1^d, L_3^d and α with respect to b_l^d can be represented in terms of E_{11}, E_{22}, v_{12} and G_{12}.

The natural frequency can now be written as follows:

$$\omega = \omega^d + \sum_{l=1}^{p}(\partial\omega^d/\partial b_l^d)b_l^r. \tag{27}$$

The variance of the natural frequency takes the form:

$$\mathrm{Var}(\omega) = E\left[\left\{\sum_{l=1}^{p}(\partial\omega^d/\partial b_l^d)b_l^r\right\}^2\right]. \tag{28}$$

The SD is obtained as the square root of the variance.

3. Results and discussion

The closed form expressions developed in the previous section have been used to obtain the frequency mean and the variance for plates with random material properties. Results have been obtained for symmetric and antisymmetric cross-ply laminates. All lamina are assumed to have the same thickness and the material properties are orthotropic along the material axes. The laminates are subjected to large amplitude vibrations. The effects of material property dispersion along with variations in aspect ratio, thickness ratio and oscillation amplitude on the frequency statistics have been studied. The approach has been validated with some results available in literature.

3.1. Validation

The validation of the present formulation is sought by comparison of results with reported literature. However, non-linear formulation is not available in literature for laminated composite plate with random material properties. Hence, comparison has been made only with the linear formulation as a special case of the present non-linear analysis. Frequency variance for linear strain–displacement relations obtained by taking $L_3 = 0$ in Eq. (26), with all basic material properties varying simultaneously, are compared with the result by Singh et al. [13]. Table 1 represents the comparison of SD of frequency for four layered [0°/90°/90°/0°]

Table 1
Comparison of frequency

SD/mean of material properties	0.05	0.10	0.15	0.20
SD(ω^2)/mean(ω^2)				
Singh [13] linear	0.045	0.091	0.138	0.180
Present study (linear)	0.044	0.088	0.133	0.177
Present study (non-linear)				
($w' = 0.3$)	0.048	0.097	0.146	0.194
($w' = 0.6$)	0.054	0.107	0.160	0.213
($w' = 0.9$)	0.058	0.115	0.172	0.229

Table 2
Mean non-linear frequency, ω_{nl}/ω_1

	Amplitude		
	$w' = 0.3$	$w' = 0.6$	$w' = 0.9$
Present work	1.18	1.61	2.16
Ref. [16]	1.22	1.63	2.18

Stacking sequence: [0°/90°/90°/0°], SSSS plate, $b/h = 100$, $a/b = 2$.

square symmetric cross-ply with thickness ratio $b/h = 10$. The mean values of the material properties used are [13]:

$$E_{22} = 10.3\,\text{GPa},\ E_{11} = 25E_{22},\ G_{12} = 0.5E_{22},\ \text{and}$$
$$\nu_{12} = 0.25,$$

where E_{11} and E_{22} denote the longitudinal and transverse elastic modulii, respectively, G_{12} is the in-plane shear modulus and ν_{12} denotes the Poisson ratio. The plate edges are simply supported and the SD of frequency is non-dimensionalized with the mean frequency. A reasonable good agreement between the two is observed. The results from the present study with non-linear formulation are also placed in the table for comparison. The effect of non-linearity is apparent with different values of the amplitude $w' = w_{max}/b$. The mean frequency increases with increase in the amplitude. This indicates a stiffening behavior of the plate with displacement.

Table 2 presents a comparison of the non-dimensionalized mean frequency with limited available results by Singh et al. [16] for [0°/90°/90°/0°] laminate with $a/b = 2$. The material properties used for the analysis are [16]: $E_{11} = 40E_{22}$, $G_{12} = 0.5E_{22}$, and $\nu_{12} = 0.25$. The reference uses direct integration method for the analysis of mean frequency whereas the present approach gives an exact solution. A reasonably good agreement between the two is observed. The approximate method returns slightly higher values having a maximum error of 3.4% that keeps reducing with increase in the oscillation amplitude.

Table 3
Mean non-linear frequency ω_{nl}/ω_1

		Thickness ratio (b/h)		
		100	50	33.33
(a) *Stacking sequence*:[0°/90°/90°/0°]				
$a/b = 1$	($w' = 0.3$)	1.082	1.021	1.009
	($w' = 0.6$)	1.300	1.082	1.037
	($w' = 0.9$)	1.602	1.178	1.082
$a/b = 2$	($w' = 0.3$)	1.204	1.054	1.024
	($w' = 0.6$)	1.680	1.204	1.095
	($w' = 0.9$)	2.265	1.421	1.204
(b) *Stacking sequence*:[0°/90°/0°/90°]				
$a/b = 1$	($w' = 0.3$)	1.097	1.025	1.011
	($w' = 0.6$)	1.351	1.097	1.044
	($w' = 0.9$)	1.695	1.209	1.097

3.2. Second order frequency statistics

The material used for the graphite/epoxy composite plate is the same as employed for generating Table 2. All the four material properties are considered as random for the analysis.

3.2.1. Mean frequency

Table 3 presents the non-dimensionalized mean non-linear frequency for different plate thickness and amplitude with simply supported edges. Influence of the scattering in the material properties on the mean frequency has been obtained by allowing the coefficient of variation to change from 0% to 20% for laminated cross-ply plates. The plates have aspect ratios $b/a = 1$ and 2 and thickness ratios $b/h = 100$, 50 and 33.33 with stacking sequences of [0°/90°/90°/0°] and [0°/90°/0°/90°]. The mean non-dimensional frequency decreases with increase in the plate thickness and increases with the oscillation amplitude. The antisymmetric laminate has higher mean stiffness compared to the symmetric laminate having higher frequencies.

3.2.2. Variance of frequency

The variation of non-dimensionalized frequency with dispersion in all the basic material properties changing simultaneously are presented in Figs. 1(a) and (b). Results with the linear formulation are also plotted for comparison with the non-linear formulation. Square symmetric and square antisymmetric four layered cross-ply with $b/h = 100$ have been examined. As amplitude increases the frequency sensitivity increases for the symmetric lay-up while it decreases for the antisymmetric case. The sensitivity of the square antisymmetric cross-ply is greater than the symmetric lay-up. Coefficients of variation for the symmetric lay-up are close for different amplitude as well as the linear formulation. This does not hold for the antisymmetric case with the linear formulation overpredicting the value.

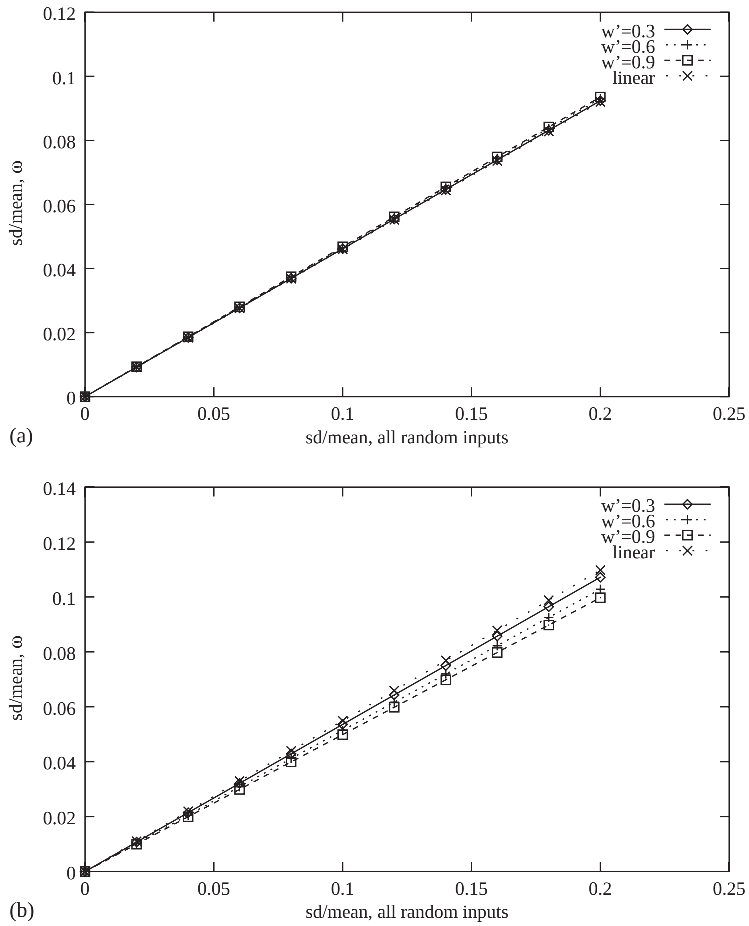

Fig. 1. Influence of SD of all basic random inputs changing simultaneously on coefficient of variation of frequency of different cross-ply with $b/h = 100$. Key: (a) [0°/90°/90°/0°] laminate with $a/b = 1$, (b) [0°/90°/0°/90°] laminate with $a/b = 1$.

Figs. 2(a)–(d) present the frequency sensitivity to dispersion with only one basic variable random at a time for square symmetric cross-ply with $b/h = 100$. The frequency variations are most affected by change in E_{11} and least affected by dispersion in v_{12}. It is also seen that sensitivity of the plate frequency to oscillation amplitude decreases with E_{22}, G_{12} and v_{12}. It, however, remains unchanged with variations in E_{11}.

It can be deduced from Eq. (27) that the frequency variance will increase with increase in the variance of the material properties. Also, the material properties with larger numerical value are expected to have greater contribution in this change. This behavior is confirmed by Fig. 1 for symmetric and antisymmetric lay-ups for all properties random and Fig. 2 for individual properties random, one at a time. However, for the square plate the reduction in SD of natural frequency with increase in oscillation amplitude for antisymmetric lay-up with all material properties random (Fig. 1b) and for symmetric lay-up with E_{22}, G_{12} and v_{12} individually random (Figs. 2b–d) is contrary to expectation as the mean frequency increases for these cases. The

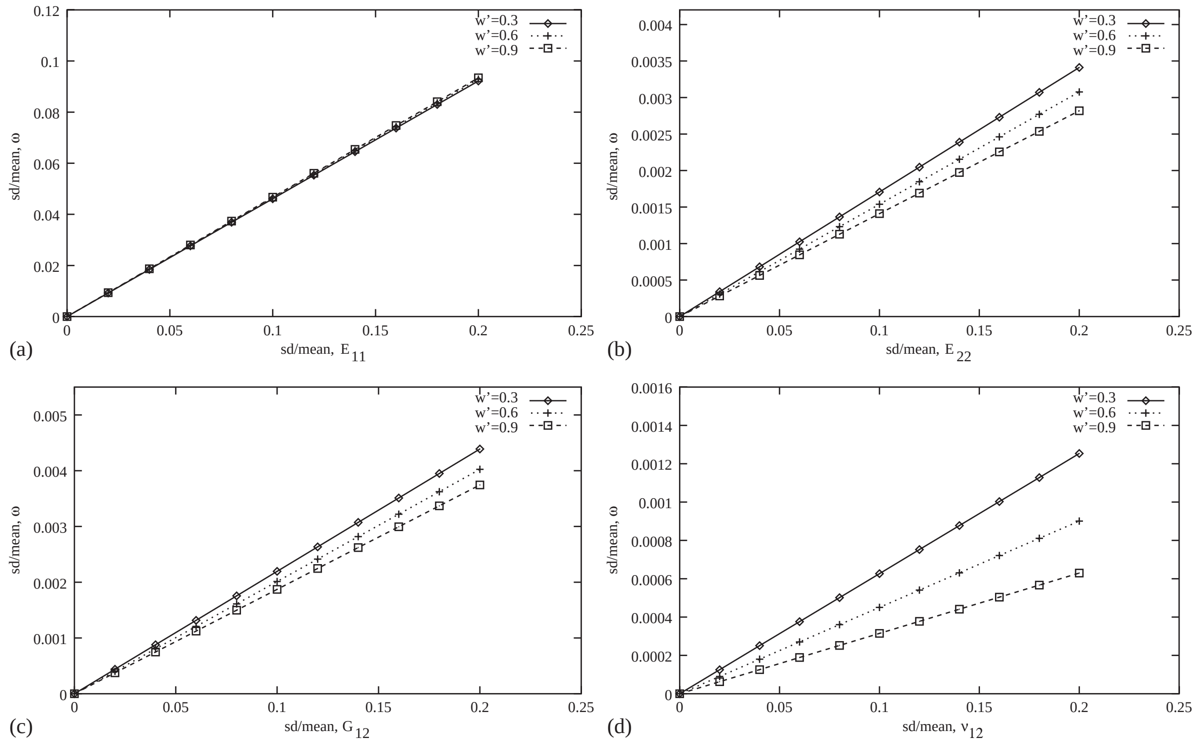

Fig. 2. Influence of SD of basic material properties on coefficient of variation of frequency of [0°/90°/90°/0°] laminate with $a/b = 1.0$ and $b/h = 100$. (a) Only E_{11} varying, (b) only E_{22} varying, (c) only G_{12} varying, (d) only ν_{12} varying.

possible reason can be that the dispersion in stiffness diminishes for these cases with increase in the oscillation amplitude. This behavior is particular to square plate only and is not exhibited in aspect ratios 2, 3 and 4 as seen in Figs. 3–5.

Frequency sensitivity to dispersion in all the basic random inputs changing simultaneously for symmetric cross-ply with aspect ratio $a/b = 1, 2, 3$ and 4 are presented in Figs. 3(a)–(d). The effect on the scattering of frequency decreases with increase in the aspect ratio from 1 to 2 and increases slightly from 2 to 3 and from 3 to 4.

For aspect ratio other than 1, as expected, the natural frequency dispersion increases with increase in the oscillation amplitude.

Figs. 4(a)–(c) show frequency sensitivity to dispersion in all the basic random inputs changing simultaneously for symmetric cross-ply with different amplitude and plate thickness ratios $(b/h) = 100$, 50 and 33.33. The effect on frequency scatter shows a decrement with increase in thickness. This is as expected as the mean frequency also decreases for these cases. This effect is less pronounced for thicker plates. As expected, the natural frequency dispersion increases with increase in the oscillation amplitude. For the linear case, frequency is independent of variations in both thickness ratios and oscillation amplitudes. The effects on frequency scatter for both square antisymmetric and rectangular symmetric laminate due to thickness ratio has the same nature as that of the symmetric cross-ply.

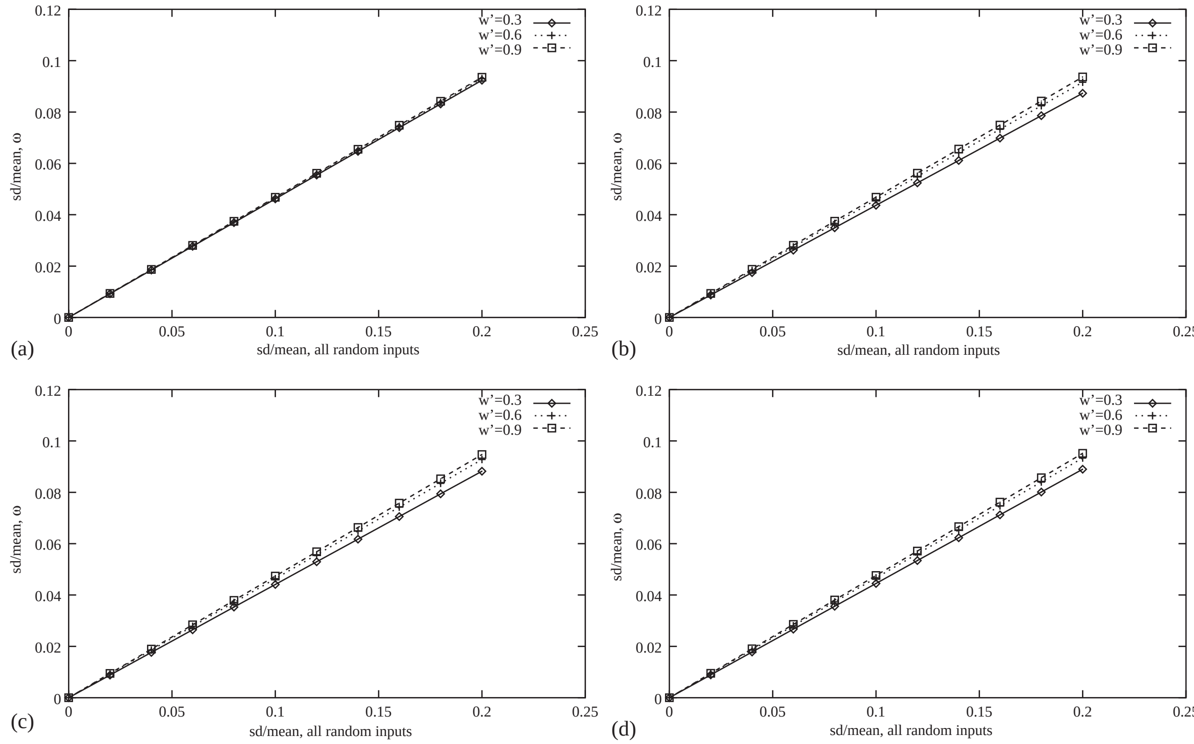

Fig. 3. Influence of SD of all basic random inputs changing simultaneously on coefficient of variation of frequency of $[0°/90°/90°/0°]$ laminate having different aspect ratio. (a) $a/b = 1$, (b) $a/b = 2$, (c) $a/b = 3$ and (d) $a/b = 4$.

Influence of amplitude $w' = w_{max}/b$ on frequency coefficient of variation with SD/mean for all material properties changing simultaneously for symmetric cross-ply with $b/h = 100$ and $b/a = 2$ is shown in Fig. 5. It is found that the increase in frequency scatter with increase in oscillation amplitude is slightly non-linear for the range considered for the study. It shows a rise with variation in material properties.

4. Conclusion

An approach has been presented to obtain the second order statistics of frequency response of laminated composite plates with random material properties for non-linear strain–displacement relations. The following conclusions have been drawn from the results for the graphite-epoxy laminated plate having all edges simply supported.

1. The influence of SD of frequency shows different sensitivity to different material properties. The sensitivity also changes with laminate construction, thickness ratios and the amplitude of oscillations.

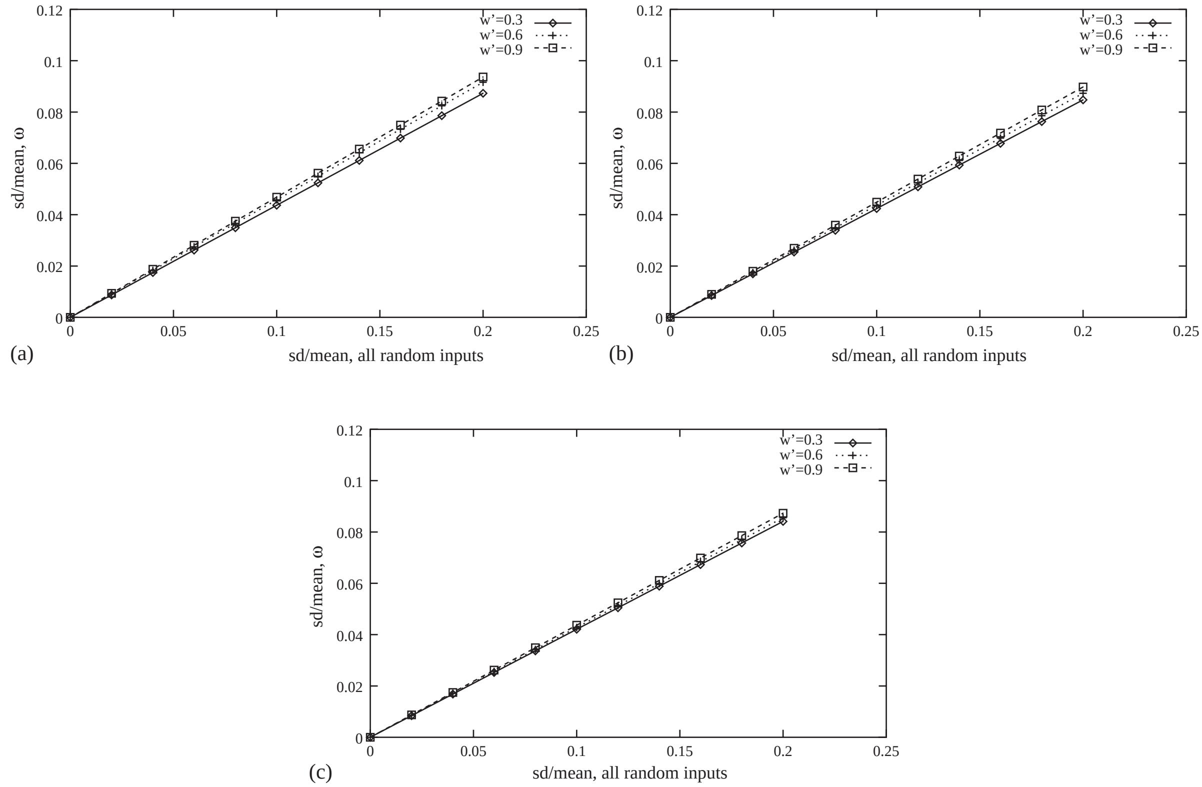

Fig. 4. Influence of SD of all basic random inputs changing simultaneously on coefficient of variation of frequency of [0°/90°/90°/0°] laminate having different thickness ratio with $a/b = 2$. (a) $b/h = 100$, (b) $b/h = 50$ and (c) $b/h = 33.33$.

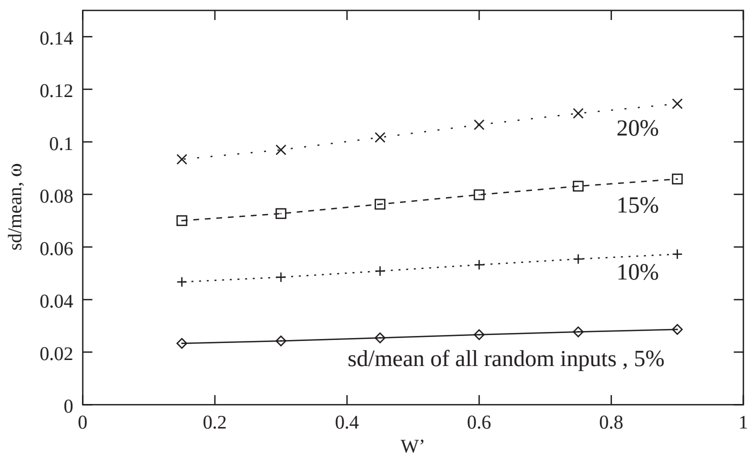

Fig. 5. Influence of amplitude on coefficient of variation of frequency of [0°/90°/90°/0°] laminate, $a/b = 2$, for all material properties changing simultaneously.

2. The dispersions in the response frequency show linear variation with SD of the material properties in the range studied.
3. Slightly non-linear variation in frequency has been seen with variation in the oscillation amplitude.
4. Antisymmetric cross-ply laminate exhibit higher sensitivity than symmetric lay-up. Increase in thickness ratio and oscillation amplitude of the plate results in increase in frequency scatter with all material properties changing simultaneously for rectangular plates. Linear analysis predicts no dependence of frequency on plate thickness ratio and oscillation amplitude.
5. The square plate shows a higher sensitivity compared to a rectangular plate. Variation of E_{11} has dominant effect on the scattering of frequency as compared to E_{22}, G_{12} and v_{12}.

Appendix A

The coefficients $T_1, T_2 \ldots, T_{10}$ as functions of plate stiffness coefficients, dimensions of the plane and mode shape are

$$T_1 = (m\pi/a)^2 A_{11} + (n\pi/b)^2 A_{66},$$

$$T_2 = (m\pi/a)(n\pi/b)(A_{12} + A_{66}),$$

$$T_3 = -(m\pi/a)^3 B_{11},$$

$$T_4 = (-\tfrac{4}{9} mn\pi^2) S_{mn}[(m\pi/a)^3 A_{11} + (m\pi/a)(n\pi/b)^2 (A_{12} - A_{66})],$$

$$T_5 = (m\pi/a)^2 A_{66} + (n\pi/b)^2 A_{22},$$

$$T_6 = -(n\pi/b)^3 B_{22},$$

$$T_7 = (-\tfrac{4}{9} mn\pi^2) S_{mn}[(n\pi/b)^3 A_{22} + (n\pi/b)(m\pi/a)^2 (A_{12} - A_{66})],$$

$$T_8 = (m\pi/a)^4 D_{11} + 2(m\pi/a)^2 (n\pi/b)^2 (D_{12} + 2D_{66}) + (n\pi/b)^4 + D_{22},$$

$$T_9 = (\tfrac{4}{3} mn\pi^2) S_{mn}[(m\pi/a)^4 B_{11} + (n\pi/b)^4 B_{22},$$

$$T_{10} = (\tfrac{9}{32})[(m\pi/a)^4 A_{11} + (n\pi/b)^4 A_{22}] + (\tfrac{1}{16})(m\pi/a)^2 (n\pi/b)^2 (A_{12} + 2A_{66}),$$

where $S_{mn} = (1 - (-1)^m)(1 - (-1)^n)$.

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